(Un)decidability of injectivity and surjectivity in one-dimensional sand automata

Gaétan Richard

Laboratoire d'informatique fondamentale de Marseille,
Aix-Marseille université, CNRS;
39 rue Joliot-Curie, 13 453 Marseille, France
gaetan.richard@lif.univ-mrs.fr

Abstract. Extension of sand pile models, one-dimensional sand automata are an intermediate discrete dynamical system between one dimensional cellular automata and two-dimensional cellular automata. In this paper, we shall study the decidability problem of global behavior of this system. In particular, we shall focus on the problem of injectivity and surjectivity which have the property of being decidable for one-dimensional cellular automata and undecidable for two-dimensional one. We prove the following quite surprising property that surjectivity is undecidable whereas injectivity is decidable. For completeness, we also study these properties on some classical restrictions of configurations (finite, periodic and bounded ones).

Introduction

Complex systems are systems made of a great number of well known entities interacting locally with each other in a fully determined way. Despite the fact that the local behavior is completely known, the global behavior of the system may be very complex and even unpredictable. One simple formal model of complex systems is cellular automata which consist on entities endowed with a state chosen among a finite set, arranged on a regular grid of fixed dimension. Dynamics of this system is obtained by applying uniformly and synchronously a local transition function. In such systems, it was proved that in for one dimensional grid, injectivity and surjectivity are decidable [1] whereas these properties are undecidable in higher dimensions [2].

Introduced as a generalisation of sand-piles models [3], sand automata [4] are a variant of cellular automata where states are integers and where the local transition function works according to the gap between the neighbour value and the cell's one. To keep some locality, the difference is said to be infinite if it excess the radius, thus the local transition function has only a finite number of cases. In a topological way, those systems can be seen as an intermediate model between one-dimensional and two-dimensional cellular automata (see [5,6]). Therefore, one natural question is the it makes sense to study decidability questions on injectivity and surjectivity in these models.

This paper is divided as follows: in section 1, we give formal definitions needed. Then in section 2, we prove undecidability of surjectivity in general as well as for classical restrictions on configurations. In section 3, we deal with the case of injectivity proving that it is decidable in general but not for all classical restrictions on configurations.

1 Definitions

In the rest of the paper, for any $a, b \in \mathbb{Z}$ with $a \leq b$ and $I \subseteq \mathbb{Z}$, let $[\![a, b]\!]$ be the set $\{a, a+1, \ldots, b-1, b\}$ and $\overline{I}$ the set $I \cup \{-\infty, +\infty\}$.

1.1 Sand automata

A *sand automaton* is a pair (r, f) where $r \in \mathbb{N}$ is the *radius* and $f : \overline{[\![-r, r]\!]}^{2r+1} \to [\![-r, r]\!]$ is the *local transition rule*. This system acts on elements $c \in \overline{\mathbb{Z}}^{\mathbb{Z}}$ called *configurations*. A configuration c is *bounded* if there exists $b \in \mathbb{N}$ such that $\forall i \in \mathbb{Z}$, $|c(i)| < b$. It is *finite* if c is constant except for a finite numbers of elements (*i.e.*, there exists $l, k \in \mathbb{Z}$, such that for all $z \in \mathbb{Z}$ such that $|z| > l$, $c(z) = k$). It is *weakly periodic* if there exists $p \in \mathbb{Z}^+, d \in \mathbb{Z}$ such that $\forall z \in \mathbb{Z}, c(z+p) = c(z) + d$. In the case where $d = 0$ in the previous definition, the configuration is called *strongly periodic*.

For any $r, l \in \mathbb{N}$, the *l-local view function* $v_l : \overline{\mathbb{Z}}^r \times \mathbb{Z} \times \overline{\mathbb{Z}}^r \to \overline{[\![-l, l]\!]}^{2r+1}$ is defined as:

$$v_l(z_{-r}, \ldots, z_0, \ldots, z_r)(i) = \begin{cases} -\infty & \text{if } (z_i - z_0) < -l \\ (z_i - z_0) & \text{if } |(z_i - z_0)| \leq l \\ +\infty & \text{if } (z_i - z_0) > l \end{cases}$$

This definition is extended to any configuration $c \in \overline{\mathbb{Z}}^{\mathbb{Z}}$ and position $z \in \mathbb{Z}$ as $v_l(c)(z) = v_l(c(z-r), \ldots, c(z+r))$ provided that $c(z) \notin \{-\infty, +\infty\}$. The *global function* $G : \overline{\mathbb{Z}}^{\mathbb{Z}} \to \overline{\mathbb{Z}}^{\mathbb{Z}}$ of a sand automaton (r, f) is defined, for all $c \in \mathbb{Z}^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ by:

$$G(c)(i) = \begin{cases} -\infty & \text{if } c(i) = -\infty \\ c(i) + f(v_r(c)(i)) & \text{if } c(i) \in \mathbb{Z} \\ +\infty & \text{if } c(i) = +\infty \end{cases}$$

An sand automaton is *injective* (resp. *surjective*) if its global function is. It is injective (resp. surjective) on finite configurations if the restriction of its global configuration to finite configurations is injective (resp. surjective). The same holds for bounded, weakly periodic or strongly periodic configurations. The links between those different properties can be found in the article of J. Cervelle, E. Formenti and B. Masson [7]. In this rest of this paper, we shall study whether those properties are decidable.

1.2 Two-counter machines

In this paper, we shall obtain undecidability result using reduction from the halting problem of two-counter machines. Let $\Upsilon = \{0, +\}$ and $\Phi = \{-, 0, +\} \times \{0, 1\}$ be respectively the set of *test values* and *counter operations*. For all $(\phi, j) \in \Phi$, *testing* $\tau : \mathbb{N}^2 \to \mathbb{N}^2$ and *modifying* $\theta : \Phi \times \mathbb{N}^2 \to \mathbb{N}^2$ actions are respectively defined for any $i \in \{0, 1\}$ and $v \in \mathbb{N}^2$ by:

$$\tau(v)_i = \begin{cases} 0 & \text{if } v_i = 0 \\ + & \text{if } v_i > 0 \end{cases} \qquad \theta(\phi)(v)_i = \begin{cases} \max(0, v_i - 1) & \text{if } \phi = (i, -) \\ v_i + 1 & \text{if } \phi = (i, +) \\ v_i & \text{otherwise} \end{cases}$$

Introduced by M. Minsky [8], *two-counter machines* (CM-2) are quadruplet (Q, q_0, q_f, t) where Q is a finite set of *states*, $q_0, q_f \in Q$ are respectively the *initial* and *final* state and $t : Q \times \Upsilon^2 \to Q \times \Phi$ the *local transition rule*. Those machines act on *configurations* $c \in Q \times \mathbb{N}^2$ by the *global transition rule* $T : Q \times \mathbb{N}^2 \to Q \times \mathbb{N}^2$ defined as $T(q, v) = (q', \theta(\phi)(v))$ when $(q', \phi) = t(q, \tau(v))$. The configuration $(q_f, (0, 0)) \in Q \times \mathbb{N}^2$ is the *halting* configuration. With these definitions, a two-counter machine is *halting* if starting from the configuration $(q_0, (0, 0))$ it eventually reaches an halting configuration. An *evolution* of a 2-CM is a sequence $(c_0, \ldots c_{n-1}) \subset (Q \times \mathbb{N}^2)^n$ where for all $i \in [\![0, n-2]\!]$, $c_{i+1} = T(c_i)$. Such evolution is *halting* if $c_0 = (q_i, (0, 0))$ and $c_{n-1} = (q_f, (0, 0))$. Thought seeming simple, this system can achieve universal computation and thus the following theorem holds:

Theorem 1 (M. Minsky [8], 1967). *The halting problem for two-counter machines is undecidable.*

2 Surjectivity

In this section, we shall reduce the previous halting problem proving undecidability of surjectivity in sand automaton.

Theorem 2. *Given a sand-automaton $S = (r, f)$, it is undecidable to know whether it is surjective.*

The reduction use the following sketch. We first define an encoding of any evolution of any CM-2 inside a configuration of a sand automaton. Then, for each two-counter machine, we define a sand automaton that is surjective on all configurations except those containing the encoding of an halting evolution. To do this, the constructed sand automaton "checks" locally whether the configuration seems to be a correct evolution of the machine. In this case, the automaton does a XOR on some additional *checking bits*. The main point is that those bits are positioned such that a valid halting evolution creates a finite cycle and thus prevents surjectivity. The idea of this technique is similar to the one used in the proof of undecidability of surjectivity over finite configurations in two-dimensional cellular automata by J. Kari [9] whereas realisation is trickier due to additional restrictions encountered.

2.1 Proof of the theorem

Encoding two-counter machine evolution. This section is devoted to explain how to encode an evolution of a two-counter machine inside a sand automaton configuration. One trick in the encoding is that all data are encoded by sequence of integers which are all multiple of 10. Intermediate values being only used to achieve unambiguity. Therefore, to ease reading and understanding, values in configurations of sand automaton are given as one digit numbers (ex: 0.5).

For any configuration $c = (q, l, r) \in Q \times \mathbb{N}^2$, any integer $h \in [\![-r-1, l+1]\!]$ and any array of *checking bits* ($x = (x_0, \ldots, x_7) \in \{0,1\}^8$), the $c - h - x$ *snapshot* is the sequence of values (depicted in Fig. 1) obtained by concatenating the following sub-sequences where $l_? = 1$ iff $l = 0$ (resp. $r_? = 1$ iff $r = 0$):

- $(l+1, l+1.3, l+1+x_0, l+1+x_1)$ to encode first counter;
- $(-r-1, -r-1+0.4, -r-1+x_2, -r-1+x_3)$ to encode second counter;
- $(0, .1, x_4, x_5)$ to encode zero;
- $(h, h+0.2, h+q, h+l_?, h+r_?, h+x_6, h+x_7)$ to encode the state.

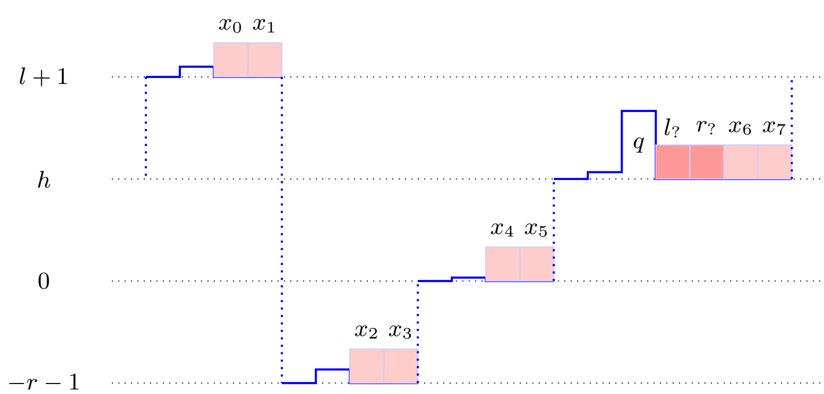


Fig. 1. A $(q, l, r) - h - x$ *snapshot*

A $c - h$ *snapshot* (denoted as $\mathcal{S}_c^h$) a $c - h - x$ snapshot for an arbitrary $x \in \{0,1\}^8$. With this notation, a configuration $c = (q, l, r)$ is encoded as any $c - 0$ snapshot and a transition between $c = (q, l, r)$ and $c' = (q', l', r')$ is encoded as following sequence of snapshots:

$$\mathcal{T}_{c,c'} = \mathcal{S}_c^1 \ldots \mathcal{S}_c^l \mathcal{S}_{(q,l',r)}^{l+1} \mathcal{S}_{(q,l',r)}^{l} \mathcal{S}_{(q,l',r)}^{l-1} \ldots \mathcal{S}_{(q,l',r)}^{-r} \mathcal{S}_{(q,l',r')}^{-r-1} \mathcal{S}_{(q,l',r')}^{-r} \ldots \mathcal{S}_{(q,l',r')}^{-1}$$

One way to depict this encoding is to represent each $\mathcal{S}_c^i$ as a four valuated function (one value for $-l-1, 0, i$ and $r+1$) and join those points leading to the figure 2.

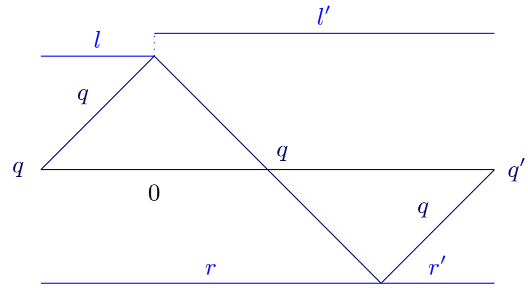


Fig. 2. Encoding a transition $(q, l, r) \vdash (q', l', r')$

With this encoding, to any 2-CM (Q, q_0, q_f, t) and any evolution $(c_0, \ldots c_{n-1}) \subset (Q \times \mathbb{N}^2)^n$, one the set $S \in \mathbb{N}^k$ of partial configuration of sand automata on the form:

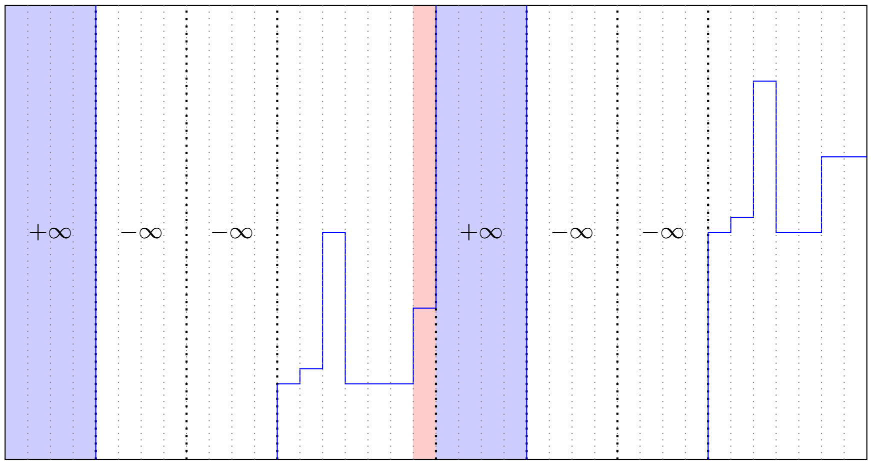
$$\mathcal{S}^0_{c_0} \mathcal{T}_{c_0,c_1} \mathcal{S}^0_{c_1} \ldots \mathcal{T}_{c_{n-2},c_{n-1}} \mathcal{S}^0_{c_{n-1}}$$

The idea is now to construct, for any 2-CM, a sand automaton which is surjective on all configuration except those containing an encoding of an halting configuration.

Construction of the automaton. The main point of the constructed sand-automaton is to "check" whether the current configuration contains the encoding of an halting evolution. To do this, we use a neigbourhood of large size which is more than twice the size of a snapshot which ensure that our local view contains at least one neighbouring snapshot if it exists. As snapshots can be of arbitrary large height, one cannot see the whole contents of the snapshot though the local view. However, using the small "bumps" (gaps between 0.1 and 0.4) in the encoding, one can determine in which section (head, stacks or zero) is the current position and whether this is compatible with the same section in the left and right neighbours.

The local transition function is chosen to be identity except for position corresponding to one of the checking bit x_i where the local view correspond to a partial correct encoding (see Fig. 3). In this case, we choose either to do a XOR with the value of the corresponding checking bit of either the left or the right neighbour. The former is denoted as x_i^- whereas the latter is denoted as x_i^+. With this definition, the sand automaton behaves as identity except for valid portions of encoding where it behaves as a one dimensional XOR.

The last point of the construction is to define which neighbour is used in each situation and add some additional rules to ensure that the line of xored bits go trough the whole encoding. Thus, we define the order of checking presented in figure 4. For example, when seeing two consecutive snapshots on the form $\mathcal{S}^c_l \mathcal{S}^{(q,l',r)}_{l+1}$ the checking bit x_0 is xored with the checking bit x_6 rather than using the next x_0 bit. Using the depicted order, the resulting line of xored



Here the local transition rule enforce the central red column to decrease by one since the local view correspond to a possibly valid encoding and the values of the x_7 in the current and next blocs are both one.

Fig. 3. Example of non-zero transition for x_7^+

bits is compound either of repetitions of the pattern $x_0^+ x_6^- x_4^- x_7^- x_1^+ x_6^+ x_4^- x_7^+$ or $x_2^- x_7^+ x_5^+ x_6^+ x_3^- x_7^- x_5^+ x_6^-$. One important thing to notice is that all data need for applying the transition of the counter machine is included in the head portion $(q, l_?, r_?)$ and thus can be read when the sequence of snapshot change the value of l or r (as for example for $\mathcal{S}_l^c \mathcal{S}_{l+1}^{(q,l',r)}$). For the case of initial or halting configuration, we add the following additional order of checking $(x_6^- x_1^+)(x_2^- x_5^+)(x_3^- x_0^+)$ which links the two previously introduced patterns.

Lemma 1. *For the constructed sand-automaton, a configuration contains a cycle of xored bits if and only if it encodes an halting evolution of the associated CM-2.*

Proof. It is clear that the encoding of a correct evolution implies the existence of a cycle of xored bits.

Let us now look at the converse and assume there exists a cycle of xored bits. The first easy remark is that the cycle is restricted by the order of x_i defined previously. Do to this choice, the cycle is made of convex polyominoes. This ensure that the cycle is compound of succession of triangles and trapezes which form a valid upper or lower part.

From this remark, it can be deduced that there exists exactly one starting and one halting configurations and that the two parts (lower and upper) are coherent. Thus, looking around checking bits involved in the cycle, there is the encoding of a valid evolution of the associated CM-2 from the initial configuration to the halting one inducing that the CM-2 is halting. $\square$

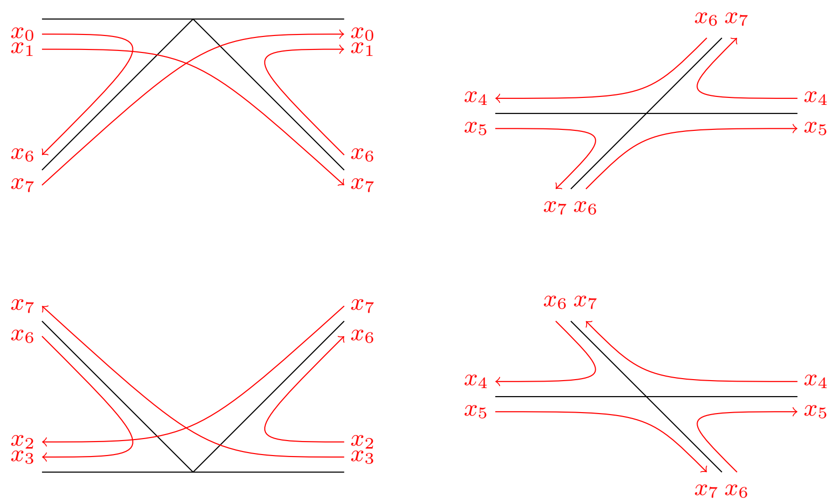


Fig. 4. Order of checks

This lemma concludes the proof of the theorem: since one dimensional XOR is surjective on finite configuration and infinite one but not for cyclic configurations of fixed size, the constructed automaton is not surjective if and only if the CM-2 is halting.

2.2 Specific cases

Using some variations of the previous construction, we can achieve to proove undecidability of surjectivity on several restrictions of configurations[1].

Proposition 1. *Given a sand-automaton $S = (r, f)$, it is undecidable to know whether it is surjective on finite configurations.*

Proof. On the one hand, in our construction, for a sufficiently large n the ∞-local view 0^{2r+1} does not encode any valid configuration. This implies that our sand automaton acts as identity on it. Therefore, the only predecessor of pattern 0^{2r+1} is 0 and, if one of the constructed automaton is surjective, it is surjective on finite configuration (note that this implication is not true in general).

On the other hand, if the constructed automaton has a cycle of xored bits, then it exists a finite configuration with this cycle. Hence, it is not surjective on finite configuration. □

Proposition 2. *Given a sand-automaton $S = (r, f)$, it is undecidable to know whether it is surjective on bounded, weakly periodic or strongly periodic configurations.*

[1] Note that some of those results can also be achieved using equivalences found in the work of J. Cervelle, E. Formenti and B. Masson [4].

Proof. For this result, it is sufficient to remark that the constructed automaton is either surjective for any of these classes of configuration if the 2-CM halts or has a strongly periodic (hence also weakly periodic) and bounded counter-example that can be constructed by repeating the non-finite portion of the finite counter example.

3 Injectivity

Now let us procced with injectivity. In a first part, we use again the previous construction to prove undecidability of injectivity over finite, bounded and strongly periodic configurations. Then, we give a full new proof for decidability of the general and weakly periodic case.

3.1 On finite, bounded and strongly periodic configurations

Proposition 3. *Given a sand-automaton, it is undecidable to know whether it is injective on finite configurations.*

Proof. If we take the previous construction, one can see that the automaton is injective unless on configurations containing infinite lines of xored bits or cycles. As previously cycles correspond to halting whereas infinite lines cannot occur in a finite configuration. □

Proposition 4. *Given a sand-automaton, it is undecidable to know whether it is injective on bounded or strongly periodic configurations.*

Proof (sketch). The basic idea of this proof is the same as the previous case, that is, to get rid of the case of infinite xored bit lines which are not cycles. To do this it is sufficient to add in our encoding a constant shift between to consecutive snapshots such that any portion (head, counters or zero) is not horizontal. With this condition, any infinite xored line is necessarily unbounded and thus cannot occur in bounded or strongly periodic configuration. □

3.2 In general and weakly periodic configuration

At this point, one could think that every property is undecidable in one-dimensional sand-automata. In fact, this is not the case and injectivity in decidable in the general case. This result is very interesting since it make the status of those two properties distinct and even make distinction inside injectivity. The rest of this section is thus devoted to prove the following theorem.

Theorem 3. *It is decidable to known whether a sand-automaton is injective.*

Proof of the theorem. The idea of the proof is to show that if a sand-automaton is not injective, then there exist a pair of weakly periodic configurations with the same image and whose perdiod can be bounded. This proof is somewhat similar to the proof in the case of one-dimensional cellular automata (see [1]).

Let us fix a sand automaton (r, f) with global transition rule G. Let us take $\delta : \mathbb{N} \to \mathbb{N}$ and $\pi : \mathbb{N} \to \mathbb{N}$ defined as $d_0 = 2r + 1$, $\pi_0 = 0$ and for all $n \in \mathbb{N}$, $\delta_{n+1} = 4\delta_n(1 + (2\pi_n + 1)^{2(2r+1)})$ and $\pi_{n+1} = 4\delta_{n+1}(2r + 1)$.

Let I be an interval of $\mathbb{Z}$, two configurations $c, c' \in \overline{\mathbb{Z}}^{\mathbb{Z}}$ are I *mutually erasable* if $G(c)_{|I} = G(c')_{|I}$ and for any sub interval $I' \subseteq I$ such that $|I'| > 2r + 1$, there exists $p \in I'$ such that $c(p) \neq c'(p)$ Inside such a pair, a position $z \in \mathbb{Z}$ is at *level* l if $c(z) \neq c(z')$, for all $i \in [\![0, l-1]\!]$, $[\![\pi_i, \pi_{i+1} - 1]\!] \cap v_\infty(c)(z) \neq \emptyset$ and $[\![\pi_i, \pi_{i+1} - 1]\!] \cap v_\infty(c')(z) \neq \emptyset$. The set of positions at level l is denoted as $\Delta_l(c, c')$.

A (x, y) *mutually erasable pattern* ((x, y)-mep) is a pair $(c, c') \in [\![a-r, b+r]\!] \to \overline{[\![c, d]\!]}$) such that $b-a < x$, $d-c < y$, $v_\infty(c)(a) = v_\infty(c')(a)$, $v_\infty(c)(b) = v_\infty(c')(b)$, $G(c)_{|[\![a,b]\!]} = G(c')_{|[\![a,b]\!]}$, $c(a) \neq c'(a)$ and $c(b) \neq c'(b)$ (see Fig. 5). Intuitively, x, y mutually erasable patterns are bounded distinct portions of configuration with the same image and such that local view is the same at extremities of each configuration. The first easy result is that such patterns can be turned into two weakly periodic configurations with the same image.

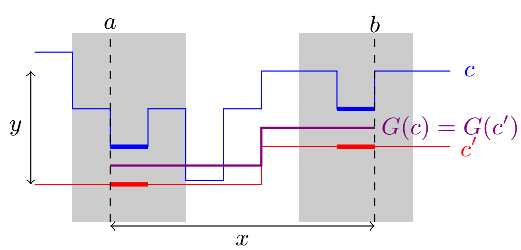


Fig. 5. Example of (x, y) mutually erasable pattern

Lemma 2. *if there exists a (x, y)-mep, then the automaton is not injective.*

Proof. Let us take (c, c') a (x, y)-mep. The basic idea is to construct a configuration by gluing successive repetitions of those patterns. To do this, let us consider the configuration $\tilde{c} : \mathbb{Z} \to \overline{\mathbb{Z}}$ defined as, for any $z \in \mathbb{Z}$,

$$\tilde{c}(z) = c(a + (z \mod (b-a))) + (G(c)(b) - G(c)(a)) \left\lfloor \frac{z}{b-a} \right\rfloor$$

This construction can also be done on c' to obtain the configuration $\tilde{c}'$. One first property is that for all $z \in \mathbb{Z}$, $\tilde{c}(z + (b-a)) = \tilde{c}(z) + (G(c)(b) - G(c)(a))$. As the same can be said on $\tilde{c}'$ and since $G(c')(b) - G(c')(a) = G(c)(b) - G(c)(a)$, it is sufficient to show that $G(\tilde{c})$ and $G(\tilde{c}')$ coincide on $[\![a, b]\!]$. However, as $v_\infty(c)(a) = v_\infty(c)(b)$,we have, for any $z \in [\![a - r, b + r]\!]$, $\tilde{c}(z) = c(z)$. The same applies for c'. Since $G(c)$ and $G(c')$ coincide on $[\![a, b]\!]$ and $c(a) \neq c'(a)$, we constructed two distinct configurations with the same image. $\square$

In the other direction, we shall prove that any non-injective sand-automata do have some mep with a computable bounded size.

Lemma 3 **(H_n).** *Let take a non-injective sand automaton then either it has a (d_n, π_n)-mep or for any I mutually erasable configuration c, c' where $|I| \geq 2\delta_n$, $\Delta_n(c, c') \neq \emptyset$.*

Proof. The case $n = 0$ is trivial since the second condition is always true.

Now, assume that H_n is true. To prove H_{n+1}, let us assume that there is no (d_n, π_n)-mep and take (c, c') two I-mutually erasable configurations with $|I| > 2\delta_n$. Without loss of generality, we can suppose that $I = [\![-\delta_n, \delta_n]\!]$ and that we have some position $p \in [\![-r, r]\!]$ such that $0 = c(p) \neq c(p')$.

The first step consist on "clipping" the configuration Let us look at the set $S = \{c(z) \mid z \in I\} \cup \{c'(z) \mid z \in I\}$. This set has at most $4\delta_{n+1}$ values thus there exists $u \in [\![1, 1 + \pi_{n+1}]\!]$ and $l \in [\![-\pi_{n+1} - 1, -1]\!]$ such that $S \cap [\![u - r, u + r]\!] = \emptyset$ and $S \cap [\![l - r, l + r]\!] = \emptyset$. Now let us construct the two elements $d, d' \in \mathbb{Z} \to [\![-\pi_{n+1}, \pi_{n+1}]\!]$ defined, for any $z \in I$, as:

$$d(z) = \begin{cases} -\infty & \text{if } c(z) < l \\ c(z) & \text{if } l \leq c(z) \leq u \\ \infty & \text{if } c(z) > u \end{cases}$$

The same can be done for d'. This operation intuitively consists on "clipping" both configurations between l and u. By construction, it can be easily seen that d and d' are I mutually erasable configurations and that $\Delta_0(d, d') \subset \Delta_0(c, c')$.

Now let us look more in details at $\Delta_0(d, d')$. By construction, we have $p \in \Delta_0(d, d')$. Since $\delta_{n+1} = 4\delta_n(1 + (2\pi_n + 1)^{2(2r+1)})$, we can divide our interval into $1 + 2(2\pi_n + 1)^{2(2r+1)}$ distinct sub-intervals of size $2\delta_n$. Now, let us prove that at least half of them do have a point at level n. The basic idea is to make use of the recurrence hypothesis and the obvious fact that a (δ_n, π_n)-mep is a $(\delta_{n+1}, \pi_{n+1})$-mep. To apply this hypothesis on any sub-interval I', we must ensure that (d, d') is I'-mutually erasable. Since (d, d') is I erasable, the fact that their image by the transition function is the same is trivial. The more difficult point is to show that $\Delta_0(d, d')$ is "dense" on the left or on the right of p. To do this, we shall proof the following lemma:

Lemma 4. *If there exists $l < p < u$ such that $\Delta_0(d, d') \cap [\![l - r, l + r]\!] = \Delta_0(d, d') \cap [\![u - r, u + r]\!] = \emptyset$ then there exists a $(u - l + 2r + 1, \pi_{n+1})$-mep.*

If we are in this conditions, one can easily obtain a mep from this by "gluing" the identical portions as depicted in figure 6. More formally, we consider the configuration e defined as:

$$e(z) = \begin{cases} d(z) & \text{if } z \in [\![l-r, u+r]\!] \\ d(z+u-l+2r) & \text{if } z \in [\![l-3r-1, l-r]\!] \\ d(z-u+l+2r) & \text{if } z \in [\![u+r, u+3r+1]\!] \\ +\infty & \text{otherwise} \end{cases}$$

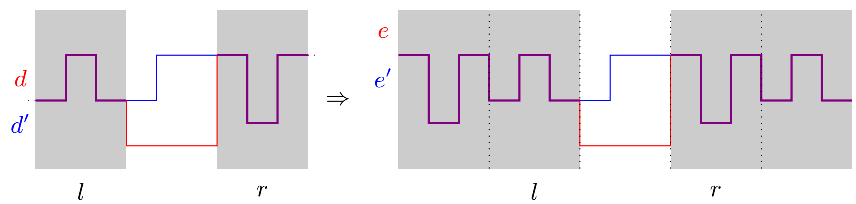


Fig. 6. Gluing by identical portion

The same can be done to obtain e' from d'. Since $d(p) \neq d'(p)$, (e, e') is a $(\delta_{n+1}, \pi_{n+1})$-mep, concluding the proof of lemma 4.

With the previous claim, we have found at least $(2\pi_n + 1)^{2(2r+1)}$ positions at level n. Since this number is more that the square the number of possible elements in v_{π_n}, there are two positions z, z' such that $v_{\pi_n}(d)(z) = v_{\pi_n}(d)(z')$ and $v_{\pi_n}(d')(z) = v_{\pi_n}(d')(z')$. If this condition would also be true for v_∞ then we would have a $(\delta_{n+1}, \pi_{n+1})$-mep. It follows that either $v_\infty(d)(z)$ or $v_\infty(d)(z')$ contains a value larger than π_n which is neither $-\infty$ nor ∞. As d has values into $[\![-\pi_{n+1}, \pi_{n+1}]\!]$, then either z or z' is at level $n+1$.

To sum up, starting for c, c' two I mutually erasable configurations with $|I| \leq 2\delta_{n+1}$ and assuming that there is no $(\delta_{n+1}, \pi_{n+1})$-mep, we have shown that there exists a point at level n (either z or z'). □

To finish the proof, it is sufficient to note that a (f, r) sand automaton cannot have any level $2r+2$ position since v_∞ contains at most $2r+1$ values and that any non injective sand-automaton has either two δ_{2r+1} mutually erasable configuration or a $(\delta_{2r+1}, \pi_{2r+1})$-mep by the same gluing argument as previously. It follows that any non-injective sand-automata have a $(\delta_{2r+1}, \pi_{2r+1})$-mep. As those mep are in finite number (up to some vertical translation), the injectivity problem is decidable for one-dimensional sand-automaton.

Conclusion

Those two results of decidability confirm the place of sand automata as an intermediate model between one and two dimensional cellular automata. The fact that status of injectivity and surjectivity differ is very interesting and could perhaps help understanding better these two notions. Even if they use the same global idea as for cellular automata, the two proofs are more subtle. The proof of undecidability of surjectivity is more powerful by working under several additional restrictions as the one for cellular automata whereas the proof of decidability of injectivity is an extension of the “cut and glue” idea used for cellular automata. For the later case, the fact that some restrictions become undecidable is also very interesting as they can all be seen as providing a way to “fix” some origin. In this way, we have somehow the same duality as between the classical halting problem and the immortality problem. It could be interesting to see if sand-automata could help provide a model where the first is undecidable whereas the second is decidable. To conclude, we can note that the bound on size of mep is rough and can probably be improved if trying to consider the complexity of deciding injectivity.

References

1. Amoroso, S., Patt, Y.N.: Decision procedures for surjectivity and injectivity of parallel maps for tessellation structures. Journal of Computer and System Sciences **6**(5) (1972) 448 – 464
2. Kari, J.: The nilpotency problem of one-dimensional cellular automata. SIAM Journal on Computing **21**(3) (1992) 571 – 586
3. Goles, E., Kiwi, M.A.: Games on line graphs and sand piles. Theoretical Computer Science **115**(2) (1993) 321–349
4. Cervelle, J., Formenti, E., Masson, B.: From sandpiles to sand automata. Theoretical Computer Science **381**(1-3) (2007) 1–28
5. Dennunzio, A., Guillon, P., Masson, B.: Stable dynamics of sand automata. Fifth Ifip International Conference On Theoretical Computer Science ,Tcs 2008 **273** (2008) 157–169
6. Dennunzio, A., Guillon, P., Masson, B.: Sand automata as cellular automata. Accepted to TCS (under revision) (2008)
7. Cervelle, J., Formenti, E., Masson, B.: Basic properties for sand automata. In: Mathematical Foundations of Computer Science 2005. (2005) 192–211
8. Minsky, M.: Computation: Finite and Infinite Machines. Prentice Hall, Englewoods Cliffs (1967)
9. Kari, J.: Reversibility and surjectivity problems of cellular automata. J. Comput. Syst. Sci. **48**(1) (1994) 149 – 182